A common framework for Minimal Length & Doubly Special Relativity

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Outlook

Considering an effective approach we can interpret the existence of a Minimal Length (ML) scale with the restriction $\Delta x_{min} > 0$.

Extending this idea to 4D plus the requirement of reference frame invariance faces us with Double Special Relativity (DSR).

Experimental observation that can be related with this ML/DSR common framework

- OPERA 11': super-luminal neutrino $\rightarrow \Delta x/\Delta t > 1$.

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1D-ML at finite order in p

- First order approximation for modified $[x, p]$ commutator

$$[x, p] = i\hbar(1 + l^2 p^2) \quad \Rightarrow \quad \Delta x \Delta p \geq \frac{\hbar}{2}\langle 1 + l^2 p^2 \rangle.$$

- Considering the limiting case

$$\Delta x(\Delta p) = \frac{\hbar}{2}(\frac{1}{\Delta p} + l^2 \Delta p) \quad \Rightarrow \quad \Delta x_0 = \hbar l.$$

- Modified action of operators on x or p space.
- If we consider usual $H(x, p)$ we obtain modified Schroedinger equations $\rightarrow$ ML phenomenology.
- From this example we can extend to N-Dimensional ML and more orders in l $\rightarrow$ ML literature.

4D-ML at every order in p

- We assume $\rho^\mu(p) = (F(p^0), p^i F(p))$ as the generator of translations in x^μ ($\hbar = c = 1$, $\eta^{\mu\nu} = diag(1, -1, -1, -1)$)

$$[x^\mu, \rho^\nu(p)] = -i\eta^{\mu\nu} \Longrightarrow [x^\mu, p^\nu] = -i\partial p^\nu / \partial \rho_\mu(p).$$

- From which we can obtain the GUPs in time and position

$$\Delta x^0 \Delta p^0 \geq \frac{1}{2}\langle \partial p^0 / \partial \rho^0(p^0)\rangle, \;\; \Delta x^i \Delta p^i \geq \frac{1}{2}\langle \partial p^i / \partial \rho^i(p)\rangle.$$

- Considering the limiting case we can derive the squeezed equations:

$$\begin{aligned}(i\partial/\partial\rho^0 + ik_0 p^0(\rho^0))\psi_T(\rho^0) &= 0 \\ (i\partial/\partial\rho + ikp(\rho))\psi_l(\rho) &= 0.\end{aligned}$$

- After solving these equations we can compute the expressions for $\Delta t(k^0)$ and $\Delta x(k)$ and minimize with respect to k^0 and k.

- From the computation of the minimal lengths for time and distance we can notice that not every $\rho^\mu(p)$ function is useful to construct a ML scenario. For example:

GOOD function	BAD function
$\rho^0(p^0) = \frac{1}{T}\tanh(Tp^0)$ $\rho^i(p^i) = \frac{p^i}{lp}\tanh(lp)$	$\rho^0(p^0) = \frac{1}{T}\ln(1+Tp^0)$ $\rho^i(p^i) = \frac{p^i}{lp}\ln(1+lp)$
$\Delta x_0 = l$, $\Delta t_0 = T$	$\Delta x_0 = 0$, $\Delta t_0 = 0$
Bounded function	Unbounded function

- Main message from 4D-ML: the functions $\rho^\mu(p)$ have to be Bounded to get a ML scenario.

Reference frame invariance

- 4D-ML is based in the commutator structure

$$[x^\mu, \rho^\nu(p)] = -i\eta^{\mu\nu} \quad \Rightarrow \quad [x^\mu, p^\nu] = -i\partial p^\nu/\partial\rho_\mu(p).$$

- What about the invariance of this structure under Lorentz transformations generated by the operator $J^{\mu\nu}$
  - Operator ρ^μ bounded $\rightarrow\leftarrow$ ρ^μ transforming as a LV.
  - $[x, \rho] = \eta$ and ρ^μ bounded $\rightarrow\leftarrow$ x^μ transforming as a LV.
  - Operator p^μ is unbounded $\rightarrow$ p^μ transforming as a LV.
- Given p^μ transforming as a LV we can find the commutator between x^μ and $J^{\mu\nu}$ using the Jacobi Identity.

$$[x^\alpha, J^{\mu\nu}] = ix^\omega \frac{\partial}{\partial\rho_\alpha}\left(p^\nu \frac{\partial\rho_\omega(p)}{\partial p_\mu} - p^\mu \frac{\partial\rho_\omega(p)}{\partial p_\nu}\right)$$

- While the transformation of the operator p^μ is as usual, the transformation of x^μ is now dependent on the momentum through the $\rho^\mu(p)$ map.
- The finite transformation of x^μ is in principle difficult to find, however we can use the previous algebra to find a function of x^μ and p^ν that transform as a LV. This function is given by:

$$x^\mu f_\mu{}^\nu(p) \text{ with } f_\mu{}^\nu(p) = \frac{\partial \rho_\mu}{\partial p_\nu}.$$

- In terms of this object we can write the modified Lorentz transformations for our system:

$$\begin{aligned} p^{\mu'} &= \Lambda^\mu{}_\nu(\beta) p^\nu \\ x^{\alpha'} f_\alpha{}^\mu(p') &= \Lambda^\mu{}_\nu(\beta) x^\alpha f_\alpha{}^\mu(p). \end{aligned}$$

- This kind of transformations belong to the general subject of Double Special Relativity.

ML/DSR transformations

- To understand the action of these DSR transformations let me consider a massive particle at rest.

$$q^\mu = (m,0,0,0) \text{ and } y^\mu = (t,0,0,0).$$

- Using the GOOD function defined for the ML scenario we can compute a DSR boost for a parameter $\vec{\beta}$, then

$$\begin{aligned} p^\mu &= (\gamma_\beta m, \gamma_\beta m\vec{\beta}) \\ v &= \frac{|\vec{x}|}{x^0} = \frac{\cosh^2(lp)}{\cosh^2(Tp^0)}\beta. \end{aligned}$$

- From these expressions we can identify two well known problems associated to DSR: the soccer ball problem and violation of locality.

- Solution: in order to implement a simple solution for both problems we choose the ML parameters T and l as given by:

$$T = 1/\alpha_T m \text{ and } l = 1/\alpha_l m.$$

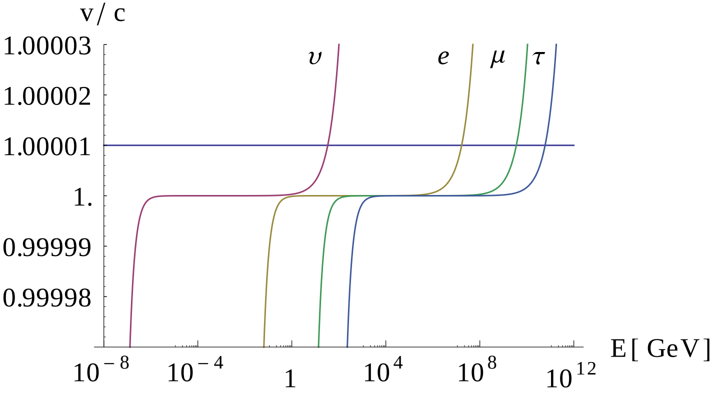
- Then $v(\beta)$ is distorted but independent of the mass $\rightarrow$ no violation of locality.
- ML/DSR effects are appreciably only for Highly Relativistic objects $\rightarrow$ no soccer ball problem.
- With this parametrization we can obtain a suggestive formula for v in terms of E, p and m

$$v = \frac{\cosh^2(\frac{p}{\alpha_l m})}{\cosh^2(\frac{E}{\alpha_T m})}\sqrt{1 - \frac{m^2}{E^2}}.$$

ML/DSR Phenomenology

- Given the previous parametrization in terms of α_T and α_l we can distinguish three cases:
  - $\alpha_T < \alpha_l \rightarrow v < \beta$ Under-luminal particles.
  - $\alpha_T = \alpha_l \rightarrow$ Allows a smooth limit $m \rightarrow 0$ (Photons).
  - $\alpha_T > \alpha_l \rightarrow v > \beta$ Potential super-luminal particles.
- Given the very NEW results from OPERA 11' about super-luminal neutrinos we are going to consider the third case.
- Writing the relation between the ML parameters as $\alpha_T^2 = 1 + \alpha_l^2$ we get $\alpha_T = 10^5 \rightarrow \Delta x_0 \sim 10^{-12}$ meters. (we are considering $m_\nu \sim 1eV$ and $v_\nu = 1.00001$)

- Using the value $\alpha_{T,l} = 10^5$ we can visualize the super-luminal behavior for neutrinos, electrons, muons and taus



Summary

- Systematic extension of ML formalism to 4D considering all orders in p.
- Considering the 4D-ML formulation we derive the associated DSR transformations.
- After implementing a simple solution for locality and the soccer ball problem the formalism is parametrized in terms of α_T and α_l.
- Considering $\alpha_T^2 = 1 + \alpha_l^2$ we get a reasonable value for Δx_0 for neutrinos and also we can accommodate a super-luminal behavior, as measured by OPERA 11'.