Moments of the Non-Diagonal Current Correlators

in QCD up to Three Loop Order

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Motivation

for this Calculation

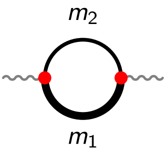
- current correlators: building blocks

  $m_1 = m_2$ heavy-heavy or diagonal
  $\sigma(e^+e^- \to \text{had.}), \Gamma(Z \to \text{had.}), \Gamma(H \to t\bar{t})$

  $m_1 \neq m_2$ heavy-light or non-diagonal
  $\sigma(q\bar{q} \to t\bar{b}), \Gamma(W^+ \to t\bar{b}), \Gamma(H^+ \to q\bar{q})$

- moments: expansion coefficients in small q^2;
  used for extraction of parameters

  - $m_{c,b}$: diagonal vector correlator (at 4L)
    $\leftrightarrow$ experiment $\sigma(e^+e^- \to \text{hadrons})$
    [Chetyrkin, Kühn, Maier, Maierhöfer, Marquard, Sturm, '07-'10]
  - $m_{c,b}$, α_s: diagonal pseudo-scalar correlator (at 4L)
    $\leftrightarrow$ moments from lattice simulations
    [Allison et al., HPQCD Collaboration, '08],
    [McNeile et al., HPQCD Collaboration, '10]



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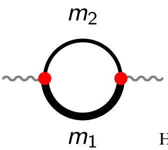
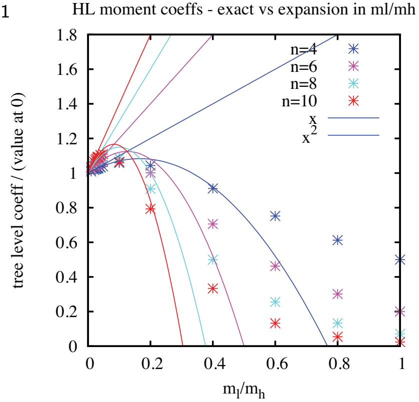


$x = m_1/m_2$

Idea

extract decay constants f_D and f_B from lattice data; study semileptonic processes [Koponen et al., HPQCD Collaboration, '10]

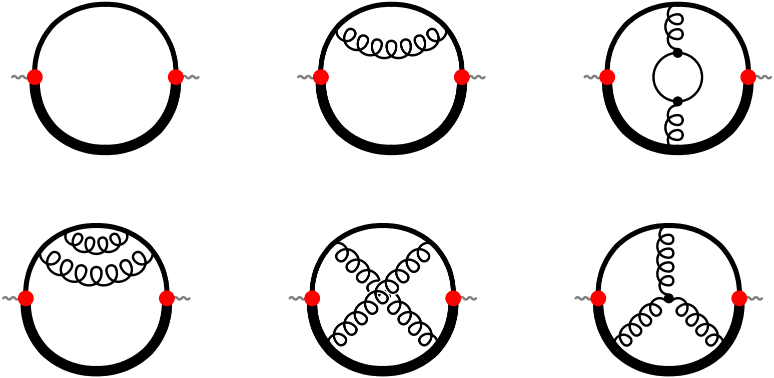
- zero-mass case not sufficient: m_c/m_b-corrections important
- lattice simulations for variety of masses (not only the physical ones)
- extrapolation to values of physical interest



Needed moments of h.-l. corr. with dependence on arbitrary masses

Sample Diagrams

at 1-, 2- and 3-Loop Order



(1, 3 and 34 diagrams)

Some Definitions

Notations & Properties

- scalar (s), pseudo-scalar (p), vector (v) and axial-vector (a) quark-currents:

$$\boxed{\begin{array}{ll} j^s = \bar{\psi}_1 \psi_2 & j^v_\mu = \bar{\psi}_1 \gamma_\mu \psi_2 \\ j^p = \bar{\psi}_1 i \gamma^5 \psi_2 & j^a_\mu = \bar{\psi}_1 \gamma_\mu \gamma^5 \psi_2 \end{array}}$$

- current correlators (polarization functions):

$$\boxed{q^2 \Pi^\delta(q^2) = i \int dx\, e^{iqx} \left\langle 0 \,\middle|\, T\, j^\delta(x) j^{\delta\dagger}(0) \,\middle|\, 0 \right\rangle} \quad (\delta = s, p)$$

$$\boxed{\begin{aligned} \left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right) \Pi^\delta_T(q^2) + q_\nu q_\mu \Pi^\delta_L(q^2) = \\ i \int dx\, e^{iqx} \left\langle 0 \,\middle|\, T\, j^\delta_\mu(x) j^{\delta\dagger}_\nu(0) \,\middle|\, 0 \right\rangle \end{aligned}} \quad (\delta = v, a)$$

Some Definitions

Notations & Properties

- dimensionless variables: $z = q^2/m_1^2$, $x = m_2/m_1$

$$\bar{\Pi}^{\delta}(q^2) = \frac{3}{16\,\pi^2} \sum_{n \geq -1} \bar{C}_n^{\delta}(x)\, z^n$$

- perturbative series:

$$\bar{C}_n^{\delta} = \bar{C}_n^{(0),\delta} + \frac{\alpha_s}{\pi} \bar{C}_n^{(1),\delta} + \left(\frac{\alpha_s}{\pi}\right)^2 \underline{\bar{C}_n^{(2),\delta}} + \mathcal{O}\left(\alpha_s^3\right)$$

Some Definitions

Notations & Properties

Checks

- $m_2 = 0$:

$$\bar{\Pi}^s = \bar{\Pi}^p \quad \text{and} \quad \bar{\Pi}^v_{T,L} = \bar{\Pi}^a_{T,L}$$

- mirror symmetry (due to γ_5):

$$\bar{C}^s_n(x) = \bar{C}^p_n(-x) \quad \text{and} \quad \bar{C}^v_n(x) = \bar{C}^a_n(-x)$$

- Ward-identity ($x = m_2/m_1$):

$$\bar{C}^v_{L,n} = (1-x)^2\,\bar{C}^s_{n+1} \quad \text{and} \quad \bar{C}^a_{L,n} = (1+x)^2\,\bar{C}^p_{n+1}$$

- at 2 loops: agreement with exact result [Djoudi, Gambino, '94/'95]
- at 3 loops: agreement with special cases $m_2 = m_1$ or $m_2 = 0$ [Chetyrkin, Kühn, Steinhauser, '97], [Chetyrkin, Steinhauser, '01]

Renormalization

of the Correlators

1. parameter-renormalization:

$$m_{1,2}^0 \rightarrow Z_m\, m_{1,2}$$
$$\alpha_s^0 \rightarrow Z_\alpha\, \alpha_s$$

2. renormalization of the current:

$$\left(j^\delta\right)_{\text{bare}} = Z_{j^\delta} \left(j^\delta\right)_{\text{ren.}} \quad \text{with} \quad \boxed{Z_{j^{s,p}} = Z_m} \quad \text{and} \quad \boxed{Z_{j^{v,a}} = 1}$$

3. QED-like overall-renormalization:

$$\boxed{\Pi^\delta(0) = 0} \quad \Longleftarrow \quad \Pi^\delta(q^2) = \bar{\Pi}^\delta(q^2) - \frac{3}{16\,\pi^2}\left(\bar{C}_0^\delta + \frac{\bar{C}_{-1}^\delta}{z}\right)$$

Details of the Calculation

of two different Expansions

- highly automated setup: QGRAF, q2e, exp and MATAD
- 3 scales $\Rightarrow$ two different expansions at 3 loops:
  - around $x = 0$: asymptotic expansion in $q^2 \ll m_2^2 \ll m_1^2$
  - around $x = 1$: naive expansion in q^2 and $\Delta m = m_2 - m_1$

$$\frac{1}{m_2 - \not{p}} = \sum_{i \geq 0} \frac{(-\Delta_m)^i}{\left(m_1 - \not{p}\right)^{i+1}}$$

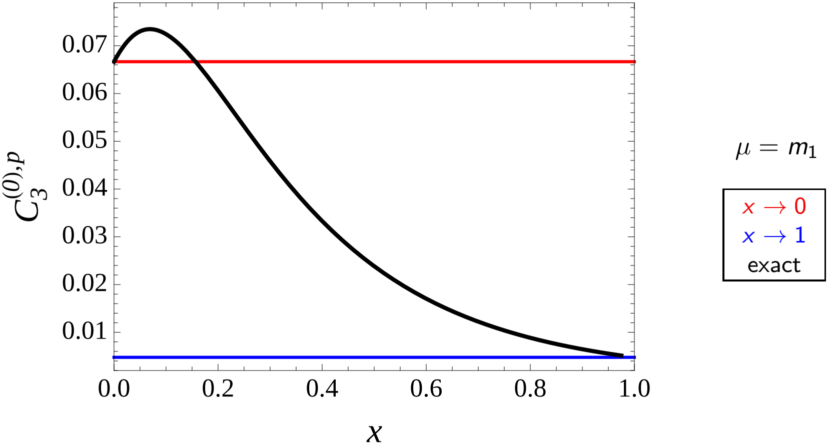
- RGE-methods: exact μ-dependence at 3 loops

$$\boxed{\mu^2 \frac{d^2}{d\mu^2} \left(Z_{j^\delta}^2 \; \Pi^\delta \right) = 0}$$

with $\quad \Pi_{(2)}^\delta = \ldots + l_\mu \underline{\ldots} + l_\mu^2 \underline{\ldots} \quad$ and $\quad l_\mu = \log \frac{\mu^2}{m_1^2}$

Results

at 1-Loop Order



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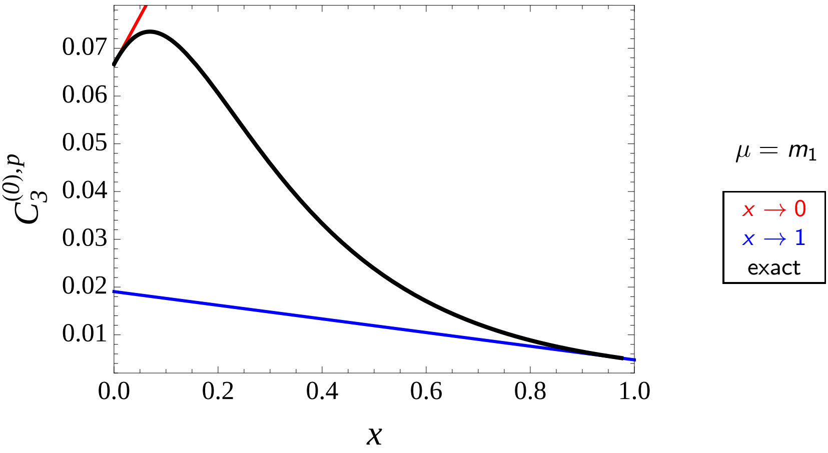

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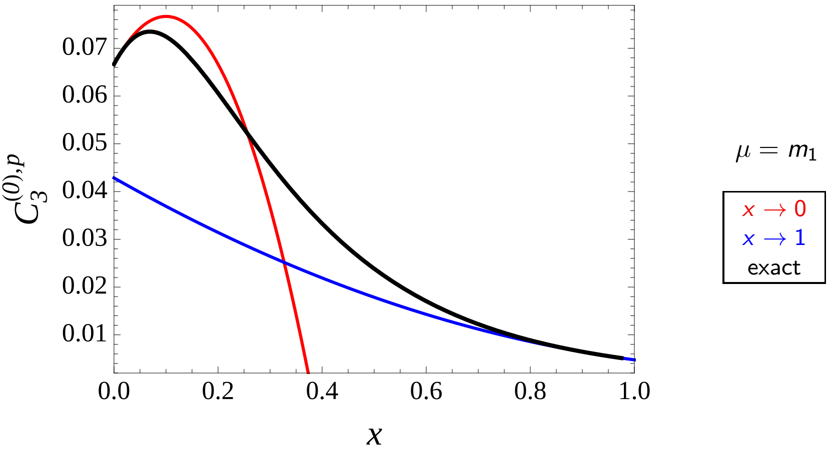

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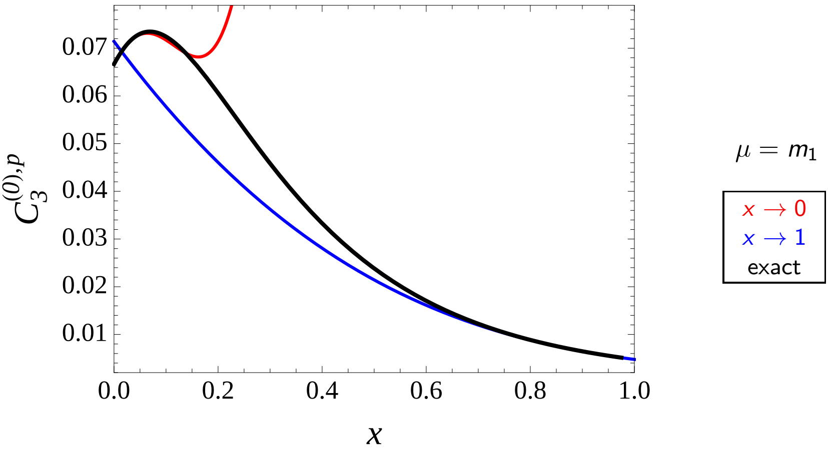

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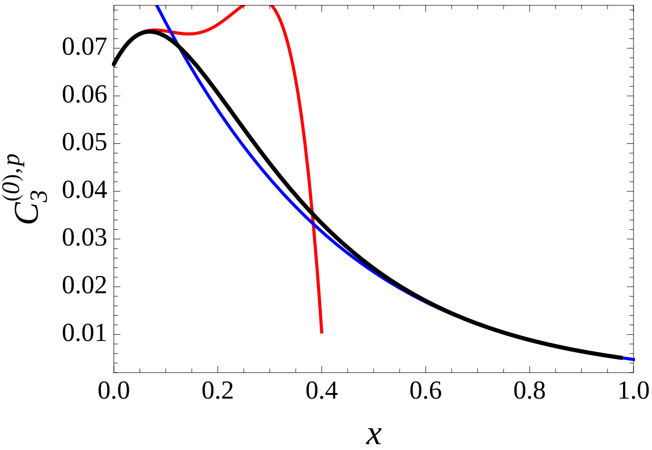
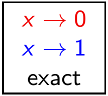


$\mu = m_1$



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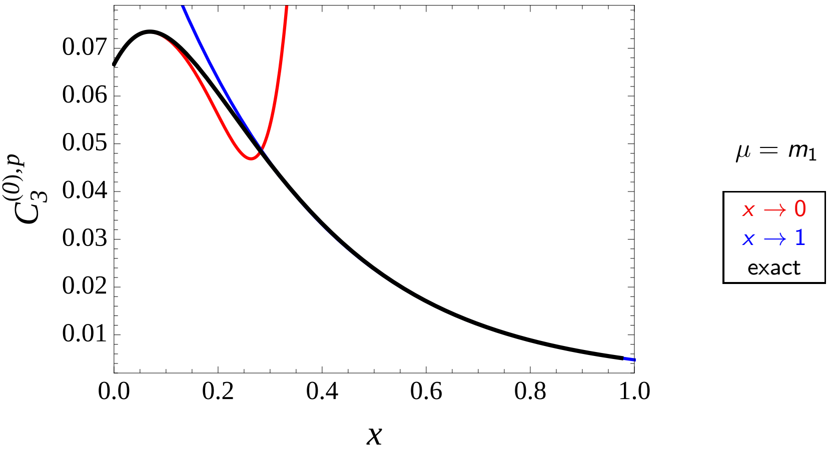

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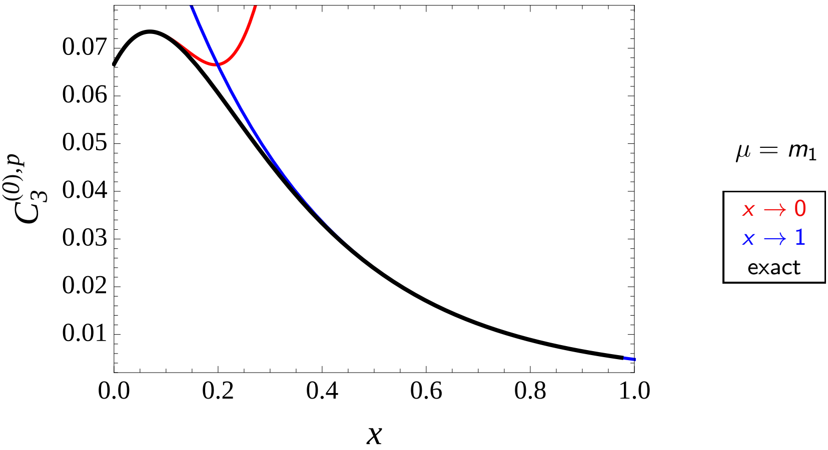

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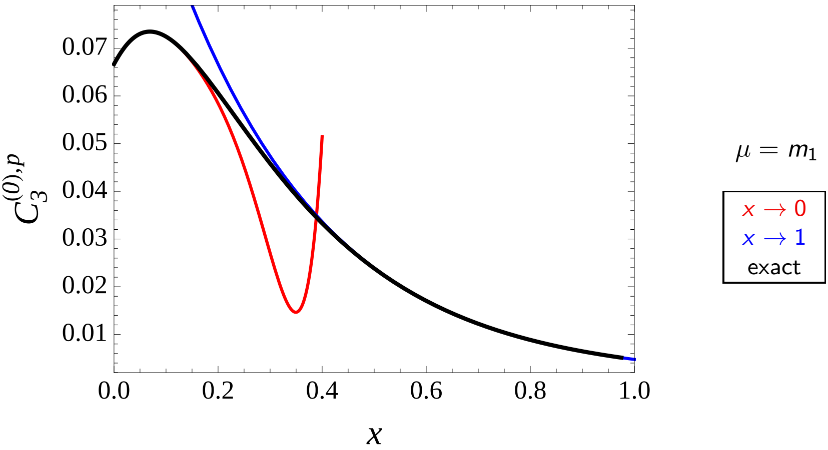

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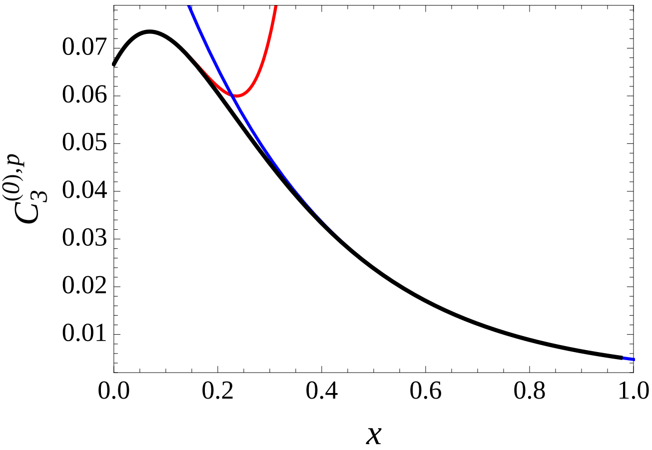
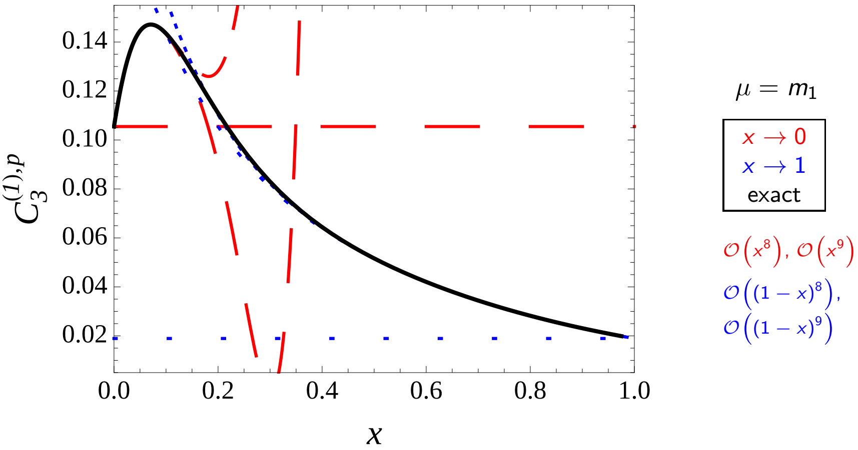


$\mu = m_1$

$x \to 0$
$x \to 1$
exact

Results

at 2-Loop Order

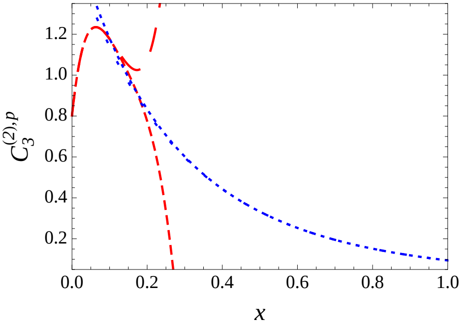


Results

at 3-Loop Order

Method

1. find x_0 and x_1 (0.1, 0.5)
2. construct interpolation function
3. fit to polynomial in $x^{1/2}$ up to $\mathcal{O}(x^3)$



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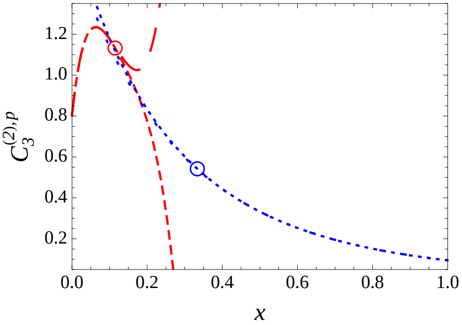

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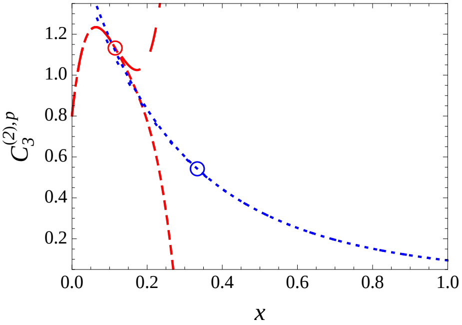

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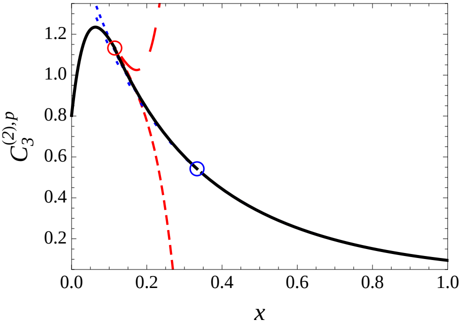

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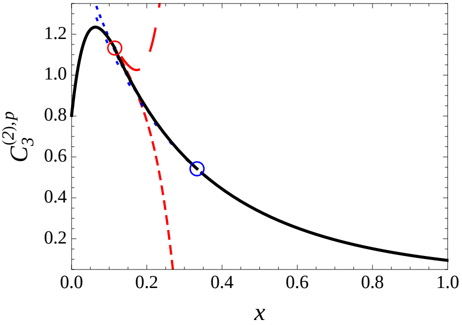


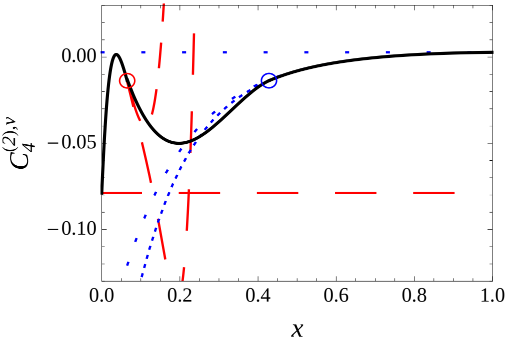
Observation

works perfectly at 1 and 2 loops $\Rightarrow$ applied at 3 loop level

Results

at 3-Loop Order

only 2 cases (out of 4 x 4 = 16) are more difficult to approximate:
$C_3^{(2),v}$ and $C_4^{(2),v}$ (more compl. behaviour)



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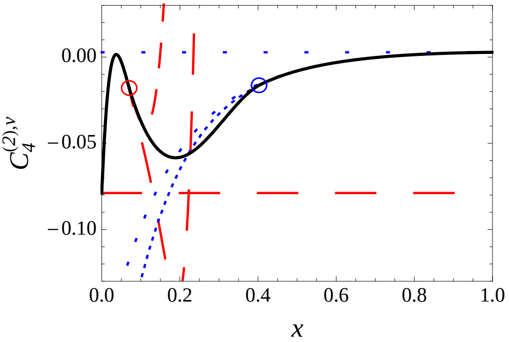

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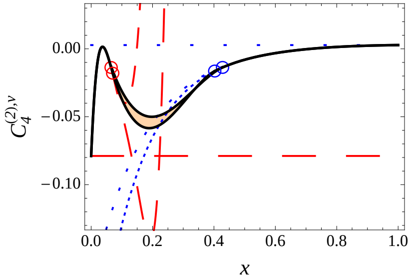


But our approach works fine in most cases
Error below 5% (estimated at 1 & 2 loops)

Summary & Outlook

Summary

- 4 cases (s, p, v, a); up to 4th moment
- 1 loop & 2 loops: exact x-dependence ($x = m_2/m_1$)
- 3 loops:
  - expansions around $x = 0$ and $x = 1$
  - combinations valid for arbitrary x
  - exact μ-dependence

$\Rightarrow$ `Mathematica` package: `coefhl.m`

Outlook

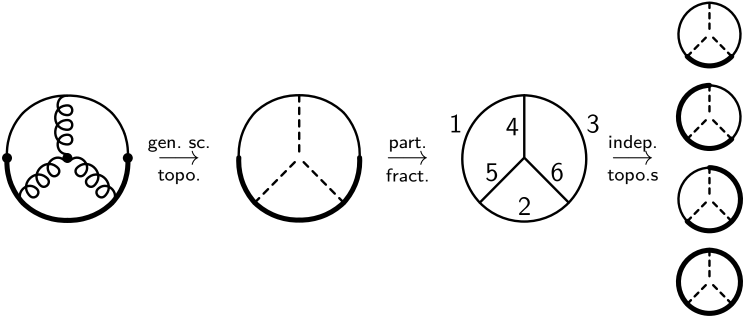
$\Rightarrow$ calculation retaining the exact x-dependence at 3 loops

$\Rightarrow$ go to higher moments

Summary & Outlook

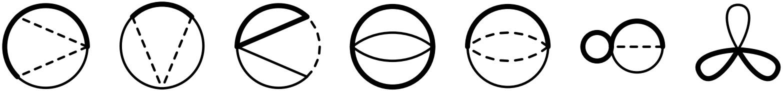
Exact Calculation

e.g.:



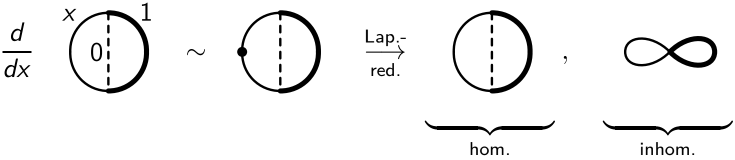
Laporta reduction to master integrals

⇓



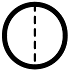
(⇒ apply Method of Differential Equations)

- 1st order diff. eqn.s (in gen.: chained): e.g.



$$\frac{d}{dx} \quad f(1,1,1) \quad \sim \quad f(1,2,1) \quad \longrightarrow \quad f(1,1,1) \quad , \quad f(1,1,0)$$

- expand in ϵ (decouple) $\Rightarrow$ solve by Euler's method using (known) 1-scale integrals as bounds; in this example

$x \rightarrow 0$:  or $x \rightarrow 1$: 

- use Harmonic Polylogs with: $f_1(x) = \frac{1}{1-x}$, $f_2(x) = \frac{1}{x}$, $f_3(x) = \frac{1}{1+x}$,

$$H(i, i_1, \ldots, i_n; x) = \int_0^x f_i(y) H(i_1, \ldots, i_n) dy$$