

Classical Dimensional Transmutation and Confinement

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Set up

$$S = \int \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} \lambda_0 \phi^4 + 4\pi Q \phi \right) d^4x$$

Static spherically symmetric solution

$$\Delta \phi - \lambda \phi^3 = -4\pi Q \delta(\mathbf{x})$$

Anti-screening

$$\phi_0 = \frac{Q}{r}$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = -4\pi Q \delta(\mathbf{x}) + \lambda_0 \frac{Q^3}{r^3}$$

If $\lambda_0 > 0 \Rightarrow$ screening, if $\lambda_0 < 0 \Rightarrow$ *anti* – screening

The perturbative expansion

$$\phi(r) = \frac{Qf(r)}{r}$$

$$f(x) = 1 + \alpha_0 e^x \int_x^\infty \left(\int_0^{x'} f^3(x'') dx'' \right) e^{-x'} dx' - N(\alpha_0),$$

$$\alpha_0 \equiv -\lambda_0 Q^2 > 0 \quad x = \ln(r/r_0)$$

$$\begin{aligned} f = & 1 + \alpha_0 x + \frac{3}{2}(\alpha_0 x)^2 + \frac{5}{2}(\alpha_0 x)^3 + \frac{35}{8}(\alpha_0 x)^4 + \frac{63}{8}(\alpha_0 x)^5 + \frac{231}{16}(\alpha_0 x)^6 \\ & + \frac{429}{16}(\alpha_0 x)^7 + \frac{6435}{128}(\alpha_0 x)^8 + \frac{12155}{128}(\alpha_0 x)^9 + \frac{46189}{256}(\alpha_0 x)^{10} + \frac{88179}{256}(\alpha_0 x)^{11} + \frac{676039}{1024}(\alpha_0 x)^{12} \\ & + \frac{1300075}{1024}(\alpha_0 x)^{13} + \frac{5014575}{2048}(\alpha_0 x)^{14} + \frac{9694845}{2048}(\alpha_0 x)^{15} + \frac{300540195}{32768}(\alpha_0 x)^{16} \\ & + \frac{583401555}{32768}(\alpha_0 x)^{17} + \frac{2267783825}{5530752}(\alpha_0 x)^{18} + \frac{418157975}{5530752}(\alpha_0 x)^{19} + \frac{34461632205}{262144}(\alpha_0 x)^{20} \dots, \end{aligned}$$

$\sqrt{1 - 2\alpha_0 x}$

$$\alpha_{eff}(x) = -\lambda_0 Q^2 f^2(x) = \alpha_0 f^2(x)$$

$$x = \ln(r/r_0)$$

$$\begin{aligned} \alpha_{eff}(x) = & \alpha_0 \left[1 + 2\tilde{x} + 4 \frac{1}{1-2\tilde{x}} + 32\tilde{x}^6 + \dots \right] \\ & + \alpha_0^2 \left[6\tilde{x} + 30 \frac{3 \ln(1-2\tilde{x})}{(1-2\tilde{x})^2} \tilde{x}^4 + \frac{4176}{5} \tilde{x}^5 + \frac{10704}{5} \tilde{x}^6 \dots \right] \\ & + \alpha_0^3 \left[4 \frac{9(\ln(1-2\tilde{x}))^2 - 9 \ln(1-2\tilde{x}) + 30\tilde{x}}{(1-2\tilde{x})^3} \frac{7248}{5} \tilde{x}^5 + 59248 \tilde{x}^6 \dots \right] + \dots \end{aligned}$$

$$\tilde{x} = \alpha_0 x$$

Renormalization group and asymptotic freedom

$$\alpha_{eff}(x) = \sum_{n=0}^{\infty} \alpha_0^{n+1} g_n(x)$$
$$x = \ln(r/r_0)$$

$$\alpha_0(r_0) \quad ??????$$

$$g_0(x) = 1, \quad g_1(x) = 2x, \quad g_2(x) = 4x^2 + 6x, \quad g_3(x) = 8x^3 + 30x^2 + 48x,$$

$$g_4(x) = 16x^4 + 104x^3 + 342x^2 + 570x,$$

$$g_5(x) = 32x^5 + 308x^4 + 1572x^3 + 4998x^2 + 8568x,$$

$$g_6(x) = 64x^6 + \frac{4176}{5}x^5 + 5880x^4 + 27612x^3 + 86832x^2 + 151956x,$$

RG properties

$$\alpha_{eff}(r) = \alpha(r_0) + \alpha^2(r_0)g_1\left(\frac{r}{r_0}\right) + \dots = \sum_{n=0}^{\infty} \alpha^{n+1}(r_0)g_n\left(\frac{r}{r_0}\right)$$

$$\frac{d}{dr_0} \left(\sum_{n=0}^{\infty} \alpha^{n+1}(r_0)g_n\left(\frac{r}{r_0}\right) \right) = 0$$

$$\frac{d\alpha(r_0)}{d \ln r_0} = \alpha^2(r_0) \frac{\sum_{k=0}^{\infty} g'_{k+1}(x)\alpha^k(r_0)}{\sum_{k=0}^{\infty} (k+1)g_k(x)\alpha^k(r_0)}$$

$$\frac{dg_{n+1}(x)}{dx} = \sum_{k=0}^n (k+1)g'_{n+1-k}(0)g_k(x)$$

In our case satisfied!!!

Gell-Mann-Low equation

$$\frac{d\alpha_{eff}(x)}{dx} = \alpha_{eff}^2(x) \sum_{k=0}^{\infty} g'_{k+1}(0) \alpha_{eff}^k(x)$$

$$\beta \equiv \frac{d\alpha_{eff}(x)}{dx}$$

$$\begin{aligned} \beta(\alpha) = \sum_{k=1}^{\infty} \beta_k \alpha^{k+1} &= 2\alpha^2 + 6\alpha^3 + 48\alpha^4 + 570\alpha^5 \\ &+ 8568\alpha^6 + 151956\alpha^7 + \dots, \end{aligned}$$

Partial resummations

$$\frac{d\alpha(x)}{dx} = 2\alpha^2(x) + 6\alpha^3 + 48\alpha^4$$

$$\begin{aligned}\alpha(x) = & \frac{\alpha_0}{1 - 2\alpha_0 x} - 3\left(\frac{\alpha_0}{1 - 2\alpha_0 x}\right)^2 \ln(1 - 2\alpha_0 x) \\ & + 9\left(\frac{\alpha_0}{1 - 2\alpha_0 x}\right)^3 \left(\ln^2(1 - 2\alpha_0 x) - \ln(1 - 2\alpha_0 x) + \frac{30}{9}\alpha_0 x\right) + O(\alpha_0^4),\end{aligned}$$

Dimensional transmutation and asymptotic freedom

$$\alpha(r) = \frac{-\lambda_0 Q^2}{1 + 2\lambda_0 Q^2 \ln(r/r_0)}$$

We can define the renormalization group invariant physical scale R_c via

$$\ln \frac{R_c}{r_0} = -\frac{1}{2\lambda_0 Q^2}$$

$$\alpha(r) \equiv -\lambda(r) Q^2 = \frac{1}{2 \ln(R_c/r)}$$



Asymptotic freedom

Beyond perturbation theory and asymptotic behavior

Exact classical β function

$$\phi = \frac{Qf(r)}{r}$$

$$f'' - f' + \alpha_0 f^3 = 0 \quad x = \ln(r/R_c)$$

$$\alpha(x) = \alpha_0 f^2(x) \quad \alpha_0 = -\lambda_0 Q^2$$

$$\beta \equiv \alpha'$$

$$\beta = 2\alpha^2 + \frac{1}{2} \left(\frac{d\beta^2}{d\alpha} - \frac{\beta^2}{\alpha} \right)$$

Weak coupling expansion and renormalons

$$\beta(\alpha) = \sum_{k=1}^{\infty} \beta_k \alpha^{k+1}$$

$$\beta_1 = 2, \quad \beta_k = \sum_{m=1}^{k-1} \left(m + \frac{1}{2}\right) \beta_{k-m} \beta_m \text{ for } k \geq 2,$$

$$\beta_1 = 2, \beta_2 = 6, \beta_3 = 48, \beta_4 = 570, \beta_5 = 8568, \beta_6 = 151956, \dots$$

$$\beta_k \simeq (k+1)! \beta_1^k$$

renormalon

"Nature" of renormalon

$$\beta = 2\alpha^2(1 + \varepsilon)$$

$$2\alpha^2 \frac{d\varepsilon}{d\alpha} = \frac{\varepsilon}{1+\varepsilon} - 3\alpha(1 + \varepsilon) = \varepsilon - 3\alpha + O(\varepsilon^2, \varepsilon\alpha)$$

$$\varepsilon(\alpha) = \left(\frac{3}{\beta_1} \text{Ei}\left(\frac{1}{\beta_1\alpha}\right) + C \right) e^{-\frac{1}{\beta_1\alpha}} + O\left(\left(e^{-\frac{1}{\beta_1\alpha}}\right)^2\right)$$

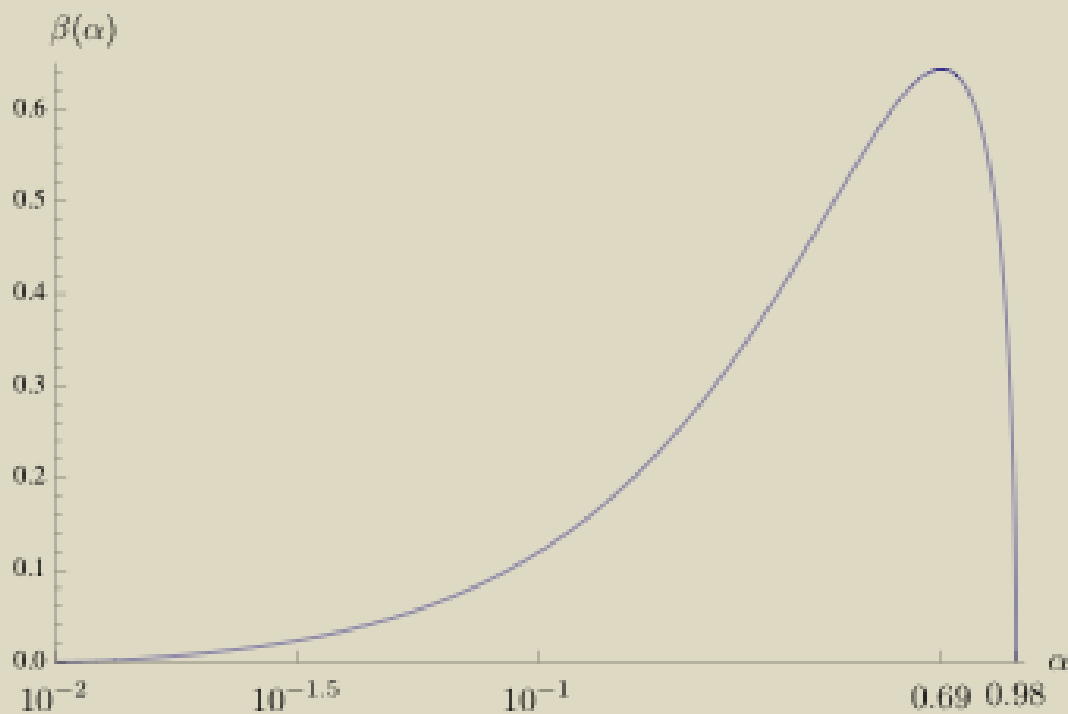
$$\text{Ei}(z) = \frac{e^z}{z} \left(\sum_{k=0}^n \frac{k!}{z^k} + O\left(\frac{1}{z^{n+1}}\right) \right)$$

$$e^{-\frac{1}{\beta_1\alpha}} \sim \frac{r}{R_c}$$

The nonperturbative solution and confinement

The infrared coupling constant

$$\frac{d\beta}{d\alpha} = \frac{2\alpha\beta - 4\alpha^3 + \beta^2}{2\alpha\beta}$$



For $\alpha > 1$ it is more convenient to work with

$$f'' - f' + \alpha_0 f^3 = 0 \quad x = \ln(r/R_c)$$

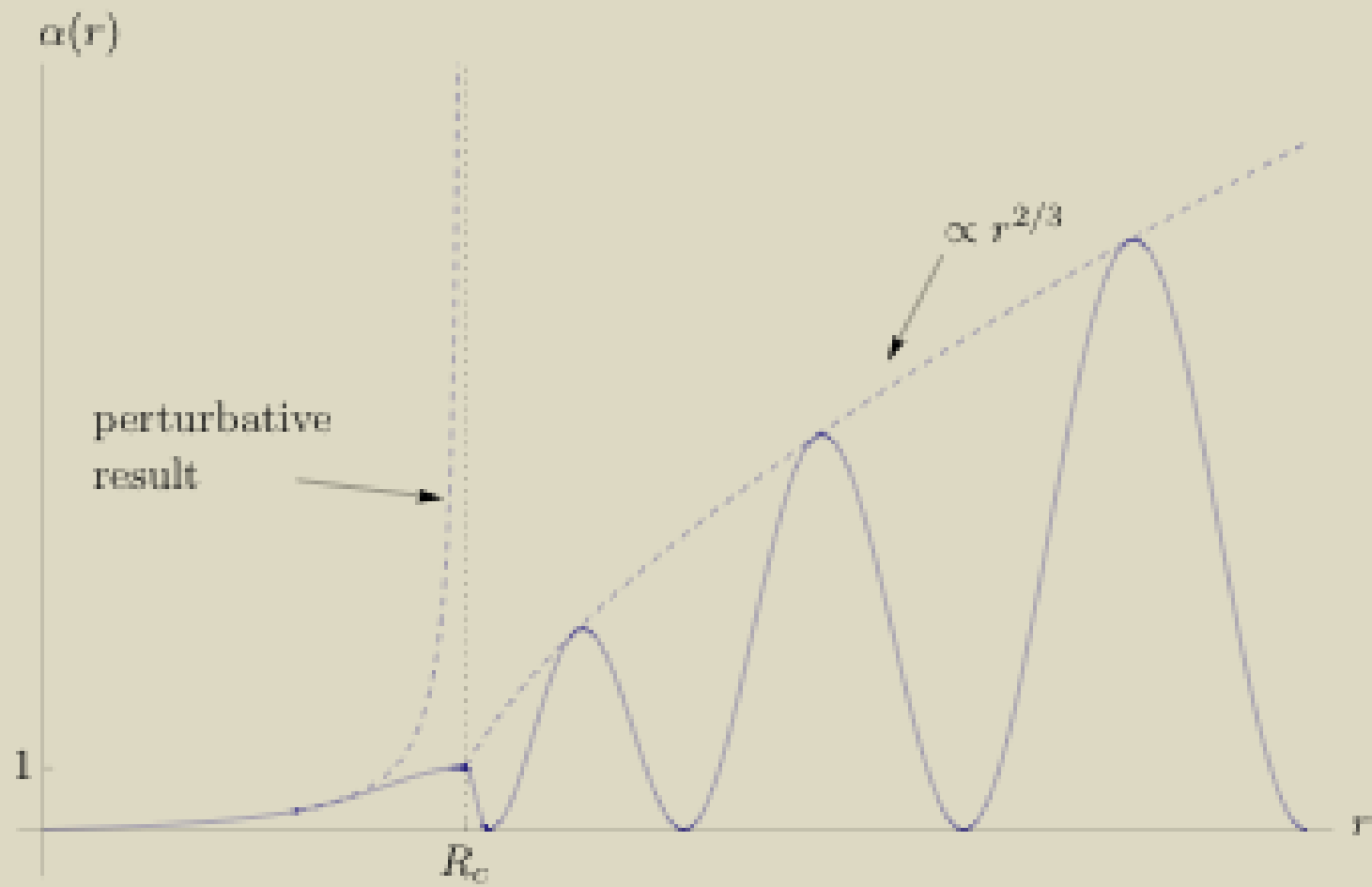
$$(ff')' - f'^2 - \frac{1}{2}(f^2)' + \alpha_0 f^4 = 0$$

$$\langle f'^2 \rangle = \alpha_0 \langle f^4 \rangle \quad \text{for } \Delta x \sim \sqrt{\frac{1}{\alpha_0 f^2}} \sim \sqrt{\frac{1}{\alpha(x)}} \ll 1$$

$$\langle f^4 \rangle = C \exp\left(\frac{4x}{3}\right) = C \left(\frac{r}{R_c}\right)^{4/3}$$

$$\alpha(r) = \alpha_0 f^2 \simeq \alpha_0 \sqrt{\langle f^4 \rangle} \cos^2\left(\int \sqrt{\alpha_0^2 \langle f^4 \rangle} dx\right)$$

$$\simeq O(1) \left(\frac{r}{R_c}\right)^{2/3} \cos^2\left(\frac{r}{R_c}\right)^{2/3}, \quad \text{for } \alpha > 1$$



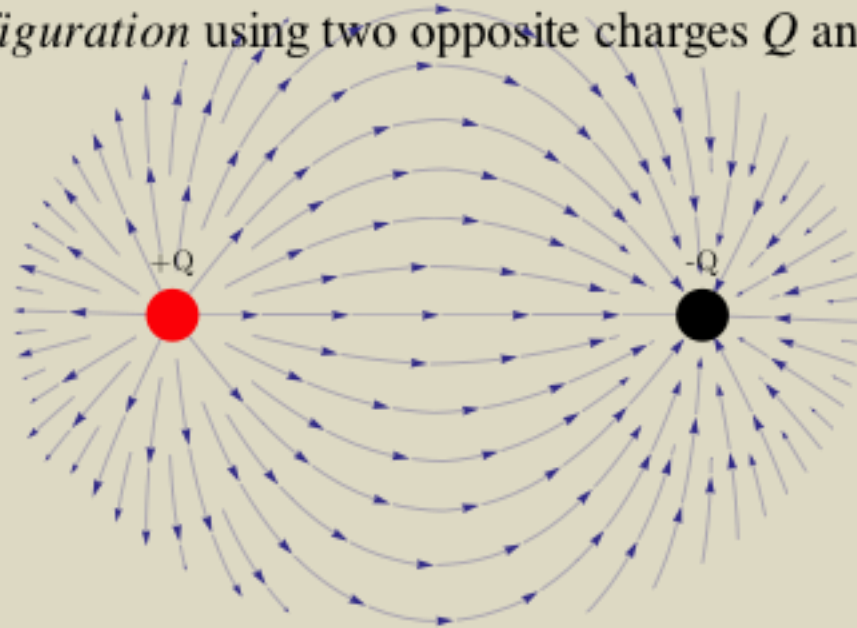
Confinement

$$V(r) \sim \frac{\alpha^2(r)}{r} \propto r^{1/3}$$

$$E = \frac{1}{2} \int \left((\nabla\phi)^2 + \frac{\lambda_0}{2} \phi^4 \right) d^3x = \pi Q^2 \int_{R_c}^r \alpha_0 \langle f^4 \rangle \frac{dr}{r^2} \sim O(1) Q^2 \frac{1}{R_c} \left(\frac{r}{R_c} \right)^{1/3}$$

The energy of the isolated charge diverges as $r^{1/3}$ and therefore it cannot exist as a free *asymptotic state*. This can be interpreted as a hint of confinement of isolated sources.

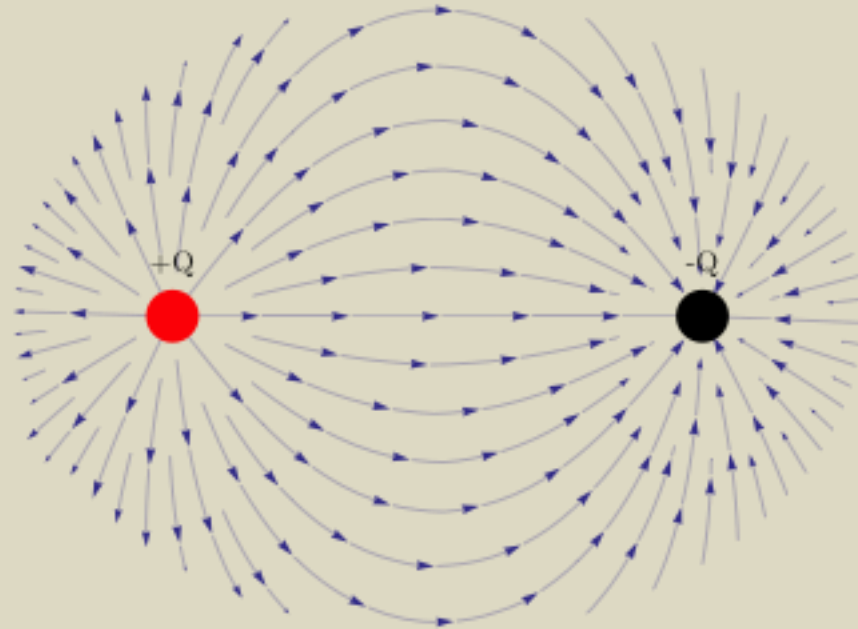
colorless configuration using two opposite charges Q and $-Q$



$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = -4\pi Q \delta(\mathbf{x}) + \lambda_0 \frac{Q^3 r^3}{r^6}$$

confinement in strong sense

$$m \sim O(1) R_c^{-1}$$



String vs. *Dipole*

On the triviality of $\lambda\varphi^4$ theory with positive λ

$$\beta = 2\alpha^2 - 6\alpha^3 + 48\alpha^4 - 570\alpha^5 + \dots$$

$$\alpha(r) = \frac{\alpha_0}{1-2\alpha_0 x} = \frac{\lambda_0 Q^2}{1-2\lambda_0 Q^2 \ln(r_0/r)}$$

$$\beta = \sqrt{2} \alpha^{3/2} \left(1 - \frac{\sqrt{2}}{3} \frac{1}{\sqrt{\alpha}} + O\left(\left(\frac{1}{\sqrt{\alpha}}\right)^2\right) \right)$$

$$\alpha(r) = 8 \left(\frac{\alpha_0}{1-2\alpha_0(x-1)} \right)^2 = 8 \left(\frac{\lambda_0 Q^2}{1-2\lambda_0 Q^2 \ln(r_0/re)} \right)^2$$

