Noncommutative Geometry in Particle Physics

or

Space-time from the Spectral Point of View

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Based on collaborative work with **Alain Connes** in publications, e.g. in

- *The Spectral Action Principle, Comm. Math. Phys. 186, 731-750 (1997)*
- *Scale Invariance in the Spectral Action, J. Math. Phys. 47, 063504 (2006)*
- *Inner Fluctuations of the Spectral Action, J. Geom. Phys. 57, 1, (2006)*
- *Gravity and the Standard Model with Neutrino Mixing, Adv. Theor. Math. Phys. 11 991-1090 (2007).*
- *Boundary Terms in Quantum Gravity from Spectral Action of Noncommutative Space, Phys. Rev. Lett. 99 071302 (2007).*
- *Why the Standard Model Journ. Geom. Phys.* 58:38-47,2008.
- *Uncanny Precision of the Spectral Action, 2009, Spectral Action for Robertson Walker metrics, 2011.*
- *A. H. Chamseddine, A. Connes, Noncommutative Geometry as a Framework for Unification of all Fundamental Interactions including Gravity. Part I , Fortsch. Phys.* ***58****, 553-600 (2010).*
- *A. H. Chamseddine, A. Connes, Space-Time from the spectral point of view, in 12th Marcel Grossmann meeting on general relativity (MG12), Editors T. Damour, R. Jantzen, and R. Ruffini, World Scientific.*
- *A. H. Chamseddine, A. Connes, Spectral Action for Robertson-Walker metrics,arXiv:1105.4637 .*

1 Introduction

We learned from General Relativity that to understand space-time we must determine its geometry. From a geometry we specify our position x in space by giving curvature invariants at x where space is modeled as a Riemannian space of dimension three. A related problem is to give observable quantities in the theory of gravity. An observable should be an invariant of the geometry. There is a "dual" point of view based on spectral invariants whose relation to the geometric one is through the heat kernel expansion of the trace of operators in Hilbert space. Our thesis is that, since much of the information we have about the nature of space-time is of spectral nature, one needs to understand carefully the process which transforms this spectral information into a geometric one. The spectrum of the Dirac operator D of a compact Riemannian space gives a sequence of invariants of the geometry: the list of the eigenvalues. It is also known that this invariant is not complete. One also needs the algebra $\mathcal{A}$ of measurable bounded functions on X acting by multiplication in the Hilbert space $\mathcal{H}$ of L^2-spinors. The spectrum of the Dirac operator D i.e. its list of eigenvalues as a subset with multiplicity inside $\mathbb{R}$ gives us the full information about the pair $(\mathcal{H}, D)$ of the Hilbert space of L^2-spinors and the Dirac operator acting in $\mathcal{H}$. Once the spectral triple $(A, \mathcal{H}, D)$ is assembled from its two pieces one recovers the points of the space and the full geometric information.

The Riemannian paradigm is based on the Taylor expansion in local coordinates of the square of the line element and in order to measure the distance between two points one minimizes the length of a path joining the two points

$$d(a,b) = \operatorname{Inf} \int_\gamma \sqrt{g_{\mu\nu}\, dx^\mu\, dx^\nu}$$

This extraction of a square root is in fact hiding a deeper understanding of the line element and the choice of a square root is associated to a global structure which is that of a spin structure. Dirac showed, in the flat case, how to extract the square root of the Laplacian in order to obtain a first order version of the Maxwell equation and Atiyah and Singer gave the general canonical definition of the Dirac operator on a Riemannian manifold endowed with a spin structure. The group $\mathcal{G}$ of symmetries of the Lagrangian of gravity coupled with matter is handed to us by physics. It is the semi-direct product of the group $\mathrm{Map}(M, G)$ of gauge transformations of second kind by the symmetry group of gravity, namely the group $\mathrm{Diff}(M)$ of diffeomorphisms of ordinary space-time M:

$$\mathcal{G} = \text{Map}(M, G) \ltimes \text{Diff}(M)$$

Now for gravity coupled with matter to be pure gravity on a new space N the most obvious requirement is to find the manifold N in such a way that

$$\text{Diff }(N) = \mathcal{G} \tag{1}$$

so one can browse through books computing diffeomorphism groups of higher dimensional manifolds N and hope for the best. The trouble is that there is no solution. This comes from a general mathematical result which asserts that the connected component of identity in Diff (N) is a *simple* group for any manifold N. Thus, since $\mathcal{G}$ has the non-trivial normal subgroup Map(M, G) there is no way one can solve the above equation using ordinary manifolds N. One can show that (1) admits a solution, provided one searches for noncommutative solutions. i.e. that the group $\mathcal{G}$ is indeed the group of diffeomorphisms of a new space N.

2 *A Brief Summary of AC NCG*

To adopt this geometrically we have to replace the notion of real variable which one takes as a function f on a set X, $f : X \to R$. It is now given by a self adjoint operator in a Hilbert space as in quantum mechanics. The space X is described by the algebra $\mathcal{A}$ of coordinates which is represented as operators in a fixed Hilbert space $\mathcal{H}$.The geometry of the noncommutative space is determined in terms of the spectral data $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \gamma)$. A real, even spectral triple is defined by

- $\mathcal{A}$ an associative algebra with unit 1 and involution $*$.
- $\mathcal{H}$ is a complex Hilbert space carrying a faithful representation π of the algebra.
- $\mathcal{D}$ is a self-adjoint operator on $\mathcal{H}$ with the resolvent $(D - \lambda)^{-1}, \lambda \notin \mathbf{R}$ of D compact.
- J is an anti–unitary operator on $\mathcal{H}$, a real structure (charge conjugation.)

- γ is a unitary operator on $\mathcal{H}$, the chirality.

We require the following axioms to hold:

- $J^2 = \epsilon$, ($\varepsilon = 1$ in zero dimensions and $\varepsilon = -1$ in 4 dimensions).
- $[a, b^o] = 0$ for all $a, b \in \mathcal{A}$, $b^o = Jb^*J^{-1}$. This is the zeroth order condition. This is needed to define the right action on elements of $\mathcal{H}$: $\zeta b = b^o\zeta$.
- $DJ = \varepsilon' JD, \quad J\gamma = \varepsilon''\gamma J, \quad D\gamma = -\gamma D$ where $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$. The reality conditions resemble the conditions of existence of Majorana (real) fermions.
- $[[D, a], b^o] = 0$ for all $a, b \in \mathcal{A}$. This is the first order condition.
- $\gamma^2 = 1$ and $[\gamma, a] = 0$ for all $a \in \mathcal{A}$. These properties allow the decomposition
  $\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R$.
- $\mathcal{H}$ is endowed with $\mathcal{A}$ bimodule structure $a\zeta b = ab^o\zeta$.
- The notion of dimension is governed by growth of eigenvalues, and may be fractals or complex.
- $\mathcal{A}$ has a well defined unitary group

$$\mathcal{U} = \{u \in \mathcal{A}; \quad u\,u^* = u^*u = 1\}$$

  The natural adjoint action of of $\mathcal{U}$ on $\mathcal{H}$ is given by $\zeta \to u\zeta u^* = u\,J\,u\,J^*\zeta \quad \forall\zeta \in \mathcal{H}$. Then

$$\langle\zeta, D\zeta\rangle$$

  is not invariant under the above transformation:

$$(u\,J\,u\,J^*)\,D\,(u\,J\,u\,J^*)^* = D + u\,[D, u^*] + J\,(u\,[D, u^*])\,J^*$$

- Then the action $\langle \zeta, D_A \zeta \rangle$ is invariant where

$$D_A = D + A + \varepsilon' JAJ^{-1}, \quad A = \sum_i a^i \left[D, b^i \right]$$

and $A = A^*$ is self-adjoint. This is similar to the appearance of the interaction term for the photon with the electrons

$$i\overline{\psi}\gamma^\mu \partial_\mu \psi \to i\overline{\psi}\gamma^\mu \left(\partial_\mu + ieA_\mu \right) \psi$$

to maintain invariance under the variations
$\psi \to e^{i\alpha(x)}\psi.$

- A real structure of KO-dimension $n \in \mathbb{Z}/8$ on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$, with the property that

$$J^2 = \varepsilon, \quad JD = \varepsilon' DJ, \quad \text{and} \quad J\gamma = \varepsilon'' \gamma J \text{ (even case)}.$$

The numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are a function of n mod 8 given by

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

- The algebra $\mathcal{A}$ is a tensor product which geometrically corresponds to a product space. The spectral geometry of $\mathcal{A}$ is given by the product rule $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$ where the algebra $\mathcal{A}_F$ is finite dimensional, and

$$\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F, \quad D = D_M \otimes 1 + \gamma_5 \otimes D_F,$$

where $L^2(M, S)$ is the Hilbert space of L^2 spinors, and D_M is the Dirac operator of the Levi-Civita spin connection on M, $D_M = \gamma^\mu (\partial_\mu + \omega_\mu)$. The Hilbert space $\mathcal{H}_F$ is taken to include the physical fermions. The chirality operator is $\gamma = \gamma_5 \otimes \gamma_F$.

In order to avoid the fermion doubling problem $\zeta, \zeta^c, \zeta^*, \zeta^{c*}$ where $\zeta \in \mathcal{H}$, are not independent) it was shown that the finite dimensional space must be taken to be of K-theoretic dimension 6 where in this case $(\varepsilon, \varepsilon', \varepsilon") = (1, 1, -1)$ (so as to impose the condition $J\zeta = \zeta$) . This makes the total K-theoretic dimension of the noncommutative space to be 10 and would allow to impose the reality (Majorana) condition and the Weyl condition simultaneously in the Minkowskian continued form, a situation very familiar in ten-dimensional supersymmetry. In the Euclidean version, the use of the J in the fermionic action, would give for the chiral fermions in the path integral, a Pfaffian instead of determinant, and will thus cut the fermionic degrees of freedom by 2. In other words, to have the fermionic sector free of the fermionic doubling problem we must make the choice

$$J_F^2 = 1, \qquad J_F D_F = D_F J_F, \qquad J_F \gamma_F = -\gamma_F J_F$$

In what follows we will restrict our attention to determination of the finite algebra, and will omit the subscript F.

3 Noncommutative Space of Standard Model

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In what follows we will restrict our attention to determination of the finite algebra, and will omit the subscript F.

- There are two main constraints on the algebra from the axioms of noncommutative geometry. We first look for involutive algebras $\mathcal{A}$ of operators in $\mathcal{H}$ such that,

$$[a, b^0] = 0\,, \quad \forall\, a, b \in \mathcal{A}\,.$$

where for any operator a in $\mathcal{H}$, $a^0 = Ja^*J^{-1}$. This is called the order zero condition. We shall assume that the following two conditions to hold. We assume the representation of $\mathcal{A}$ and J in $\mathcal{H}$ is *irreducible.*

- Classify the irreducible triplets $(\mathcal{A}, \mathcal{H}, J)$.

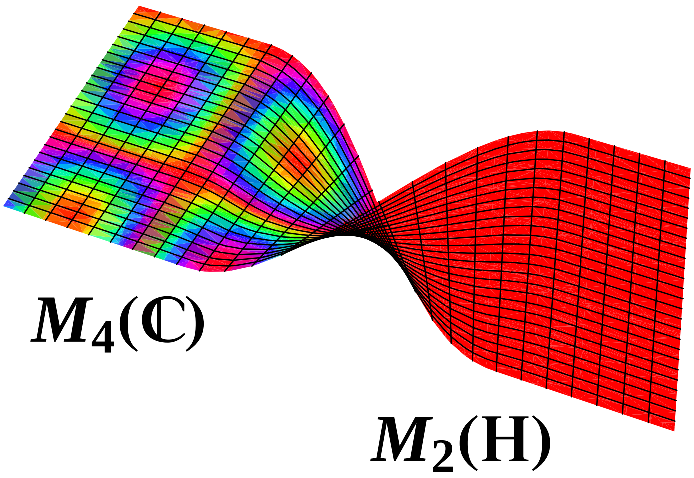
- In this case we can state the following theorem: *The center $Z(\mathcal{A}_{\mathbb{C}})$ is C or $C \oplus C$.*

- If the center $Z(\mathcal{A}_{\mathbb{C}})$ is $\mathbb{C}$ then $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$ and $\mathcal{A} = M_k(\mathbb{C})\,,\ M_k(\mathbb{R})$ and $M_a(\mathbb{H})$ for even $k = 2a$, where $\mathbb{H}$ is the field of quaternions. These correspond respectively to the unitary, orthogonal and symplectic case. The dimension of $\mathcal{H}$ Hilbert spac is $n = k^2$ is a square and $J(x) = x^*, \quad \forall x \in M_k(\mathbb{C})\,.$

- If the center $Z(\mathcal{A}_{\mathbb{C}})$ is $\mathbb{C} \oplus \mathbb{C}$ then we can state the theorem: *Let H be a Hilbert space of dimension n. Then an irreducible solution with $Z(\mathcal{A}_{\mathbb{C}}) = C \oplus C$ exists iff $n = 2k^2$ is twice a square. It is given by $A_{\mathbb{C}} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$ acting by left multiplication on itself and antilinear involution*

$$J(x, y) = (y^*, x^*)\,, \quad \forall x, y \in M_k(\mathbb{C})\,.$$

With each of the $M_k(\mathbb{C})$ in $\mathcal{A}_{\mathbb{C}}$ we can have the three possibilities $M_k(\mathbb{C})\,, M_k(\mathbb{R})\,,$ or $M_a(\mathbb{H})\,,$ where $k = 2a$. At this point we make the *hypothesis* that we are in the "symplectic–unitary" case, thus restricting the algebra $\mathcal{A}$ to the form $\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C})\,,\ k = 2a$. The dimension of the Hilbert space $n = 2k^2$ then corresponds to k^2 fundamental fermions, where $k = 2a$ is an even number. The first possible value for k is 2 corresponding to a Hilbert space of four fermions and an algebra $\mathcal{A} = \mathbb{H} \oplus M_2(\mathbb{C})$. The existence of quarks rules out this possibility. The next possible value for k is 4 predicting the number of fermions to be 16.



Up to an automorphisms of A^{ev}, there exists a unique involutive subalgebra $A_F \subset A^{ev}$ of maximal dimension admitting off-diagonal Dirac operators

$$\begin{aligned} \mathcal{A}_F &= \{\lambda \oplus q, \lambda \oplus m \,|\, \lambda \in \mathbb{C}, q \in \mathbb{H}, m \in M_3(\mathbb{C})\} \\ &\subset \mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C}) \end{aligned}$$

isomorphic to $C \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.

We denote the spinors as follows

$$\begin{aligned} \psi_A &= \psi_{\alpha I} = (\psi_{\alpha 1}, \psi_{\alpha i}) \\ &= \left(\psi_{\dot{1}1}, \psi_{\dot{2}1}, \psi_{a1}, \psi_{\dot{1}i}, \psi_{\dot{2}i}, \psi_{ai}\right) \\ &\equiv (\nu_R, e_R, l_a, u_{Ri}, d_{Ri}, q_{ai}) \end{aligned}$$

where $l_a = (\nu_L, e_L)$ and $q_{ai} = (u_{Li}, d_{Li})$. The component $\psi_{\dot{1}'1'} = \psi^c_{\dot{1}1}$ so that we get

$$\psi^*_A D^B_A \psi_B + \nu^{*c}_R k^{*\nu_R} \nu_R + cc$$

Needless to say the term $\psi^*_A D^B_A \psi_B$ contains all the fermionic interaction terms in the standard model.

Write the Dirac operator in the form

$$D = \begin{pmatrix} D^B_A & D^{B'}_A \\ D^B_{A'} & D^{B'}_{A'} \end{pmatrix},$$

where

$$A = \alpha I, \quad \alpha = 1, \cdots, 4, \quad I = 1, \cdots, 4$$
$$A' = \alpha' I', \quad \alpha' = 1', \cdots, 4', \quad I = 1', \cdots, 4'$$

Thus $D_A^B = D_{\alpha I}^{\beta J}$. We start with the algebra

$$\mathcal{A} = M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$$

and write

$$a = \begin{pmatrix} X_\alpha^\beta \delta_I^J & 0 \\ 0 & \delta_{\alpha'}^{\beta'} Y_{I'}^{J'} \end{pmatrix}$$

In this form

$$a^o = J a^* J^{-1} = \begin{pmatrix} \delta_\alpha^\beta Y_I^{tJ} & 0 \\ 0 & X_{\alpha'}^{*\beta'} \delta_{I',}^{J'} \end{pmatrix}$$

and clearly satisfy $[a, b^o] = 0$. Symplectic isometry constrains $M_4(\mathbb{C})$ to $M_2(\mathbb{H})$ and chhirality reduces this further to $\mathbb{H} \oplus \mathbb{H}$. The order one condition is

$$[[D, a], b^o] = 0$$

We have shown that the only solution of this condition is the one that breaks $\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$ to

$$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

because $X_{\dot{1}}^{\dot{1}} = Y_{1'}^{1'}$.

With this we can form the Dirac operator of the product space of this discrete space times a four-dimensional Riemannian manifold

$$D = D_M \otimes 1 + \gamma_5 \otimes D_F$$

Since D_F is a 32×32 matrix tensored with the 3×3 matrices of generation space, D is 384×384 matrix.

Next we have to evaluate the operator

$$D_A = D + A + JAJ^{-1}$$

where

$$A = \sum a [D, b]$$

or in tensor notation

$$A_A^B = \sum a_A^C \left(D_C^D b_D^B - b_C^D D_D^B \right)$$

(there are no mixing terms like $D_C^{D'} b_{D'}^B$ because b is block diagonal).

Writing all components of the the full Dirac operator $D_{\alpha I}^{\beta J}$

$$(D)_{\dot{1}1}^{\dot{1}1} = \gamma^\mu \otimes D_\mu \otimes 1_3, \quad D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{cd}(e)\,\gamma_{cd}, \quad 1_3 = \text{generations}$$

$$(D)_{\dot{1}1}^{a1} = \gamma_5 \otimes k^{*\nu} \otimes \epsilon^{ab} H_b \qquad k^\nu = 3\times 3 \text{ neutrino mixing matrix}$$

$$(D)_{\dot{2}1}^{\dot{2}1} = \gamma^\mu \otimes (D_\mu + i g_1 B_\mu) \otimes 1_3$$

$$(D)_{\dot{2}1}^{a1} = \gamma_5 \otimes k^{*e} \otimes \overline{H}^a$$

$$(D)_{a1}^{\dot{1}1} = \gamma_5 \otimes k^\nu \otimes \epsilon_{ab} \overline{H}^b$$

$$(D)_{a1}^{\dot{2}1} = \gamma_5 \otimes k^e \otimes H_a$$

$$(D)_{a1}^{b1} = \gamma^\mu \otimes \left(\left(D_\mu + \frac{i}{2} g_1 B_\mu \right) \delta_a^b - \frac{i}{2} g_2 W_\mu^\alpha \left(\sigma^\alpha\right)_a^b \right) \otimes 1_3, \qquad \sigma^\alpha = \text{Pauli}$$

$$(D)_{\dot{1}i}^{\dot{1}j} = \gamma^\mu \otimes \left(\left(D_\mu - \frac{2i}{3} g_1 B_\mu \right) \delta_i^j - \frac{i}{2} g_3 V_\mu^m \left(\lambda^m\right)_i^j \right) \otimes 1_3, \qquad \lambda^i = \text{Gell-Mann}$$

$$(D)_{\dot{1}i}^{aj} = \gamma_5 \otimes k^{*u} \otimes \epsilon^{ab} H_b \delta_i^j$$

$$(D)_{\dot{2}i}^{\dot{2}j} = \gamma^\mu \otimes \left(\left(D_\mu + \frac{i}{3} g_1 B_\mu \right) \delta_i^j - \frac{i}{2} g_3 V_\mu^m \left(\lambda^m\right)_i^j \right) \otimes 1_3$$

$$(D)_{\dot{2}i}^{aj} = \gamma_5 \otimes k^{*d} \otimes \overline{H}^a \delta_i^j$$

$$(D)_{ai}^{bj} = \gamma^\mu \otimes \left(\left(D_\mu - \frac{i}{6} g_1 B_\mu \right) \delta_a^b \delta_i^j - \frac{i}{2} g_2 W_\mu^\alpha \left(\sigma^\alpha\right)_a^b \delta_i^j - \frac{i}{2} g_3 V_\mu^m \left(\lambda^m\right)_i^j \delta_a^b \right) \otimes 1_3$$

$$(D)_{ai}^{\dot{1}j} = \gamma_5 \otimes k^u \otimes \epsilon_{ab} \overline{H}^b \delta_i^j$$

$$(D)_{ai}^{\dot{2}j} = \gamma_5 \otimes k^d \otimes H_a \delta_i^j$$

$$(D)_{\dot{1}1}^{\dot{1}'1'} = \gamma_5 \otimes k^{*\nu_R} M_R$$

$$(D)_{\dot{1}'1'}^{\dot{1}1} = \gamma_5 \otimes k^{\nu_R} M_R$$

$$D_{A'}^{B'} = \overline{D}_A^B$$

where the matrix form would look like

$$\begin{pmatrix} \dot{1}1 & \dot{2}1 & a1 & \dot{1}i & \dot{2}i & ai \\ v_R & e_R & l_a & u_{iR} & d_{iR} & q_{iL} \end{pmatrix}$$

$$\begin{pmatrix} \dot{1}1 \\ \dot{2}1 \\ b1 \\ \dot{1}j \\ \dot{2}j \\ bj \end{pmatrix} \begin{pmatrix} (D)^{\dot{1}1}_{\dot{1}1} & 0 & (D)^{a1}_{\dot{1}1} & 0 & 0 & 0 \\ 0 & (D)^{\dot{2}1}_{\dot{2}1} & (D)^{a1}_{\dot{2}1} & 0 & 0 & 0 \\ (D)^{\dot{1}1}_{b1} & (D)^{\dot{2}1}_{b1} & (D)^{b1}_{a1} & 0 & 0 & 0 \\ 0 & 0 & 0 & (D)^{\dot{1}i}_{\dot{1}j} & 0 & (D)^{ai}_{\dot{1}j} \\ 0 & 0 & 0 & 0 & (D)^{\dot{2}i}_{\dot{2}j} & (D)^{ai}_{\dot{2}j} \\ 0 & 0 & 0 & (D)^{\dot{1}i}_{bj} & (D)^{\dot{2}i}_{bj} & (D)^{ai}_{bj} \end{pmatrix}$$

4 *The Spectral Action Principle (SAP)*

- There is a shift of point of view in NCG similar to Fourier transform, where the usual emphasis on the points on the points $x \in M$ of a geometric space is now replaced by the spectrum Σ of the operator D. The existence of Riemannian manifolds which are isospectral but not isometric shows that the following hypothesis is stronger than the usual diffeomorphism invariance of the action of general relativity

The physical action depends only on the Σ

This is the spectral action principle (SAP) . The spectrum is a geometric invariant and replaces diffeomorphism invariance.

- Apply this basic principle to the noncommutative geometry defined by the spectrum of the standard model to show that the dynamics of all the interactions, including gravity is given by the spectral action

$$\text{Trace } f\left(\frac{D_A}{\Lambda}\right) + \frac{1}{2}\langle J\psi, D_A\psi\rangle$$

  where f is a test function, Λ a cutoff scale and ψ represents the fermions.

- The function f only plays a role through its momenta f_0, f_2, f_4 where

$$f_k = \int_0^\infty f(v)v^{k-1}dv, \quad \text{for } k > 0, \quad f_0 = f(0).$$

  These will serve as three free parameters in the model. $S_\Lambda[D_A]$ is the number of eigenvalues λ of D_A counted with their multiplicities such that $|\lambda| \leq \Lambda$.

To illustrate how this comes, expand the function f in terms of its Laplace transform

$$\mathrm{Trace} f\left(P\right) = \sum_{s} f_{s'} \mathrm{Trace}\left(P^{-s}\right)$$

$$\mathrm{Trace}\left(P^{-s}\right) = \frac{1}{\Gamma\left(s\right)} \int_{0}^{\infty} t^{s-1} \mathrm{Trace}\left(e^{-tP}\right) dt \qquad \mathrm{Re}\left(s\right) \geq 0$$

$$\mathrm{Trace}\left(e^{-tP}\right) \simeq \sum_{n\geq 0} t^{\frac{n-m}{d}} \int_{M} a_n\left(x, P\right) dv\left(x\right)$$

Gilkey gives generic formulas for the Seeley-deWitt coefficients $a_n\left(x, P\right)$ for a large class of differential operators P.

- The bosonic part gives an action that unifies gravity with $SU(2) \times U(1) \times SU(3)$ Yang-Mills gauge theory, with a Higgs doublet ϕ and spontaneous symmetry breaking. It is given by

$$\begin{aligned} S = \int \Big(& \tfrac{1}{2\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \gamma_0 + \tau_0 R^* R^* \\ + & \tfrac{1}{4} G^i_{\mu\nu} G^{\mu\nu i} + \tfrac{1}{4} F^{\alpha}_{\mu\nu} F^{\mu\nu\alpha} + \tfrac{1}{4} B_{\mu\nu} B^{\mu\nu} \\ + & \tfrac{1}{2} |D_\mu H|^2 - \mu_0^2 |H|^2 - \xi_0 R |H|^2 + \lambda_0 |H|^4 \Big) \sqrt{g}\, d^4x, \end{aligned}$$

  We have the following predictions:

- The gauge symmetry $U(1)\times SU(2)\times SU(3)$ comes out in almost unique way. .
- 4^2= 16 fermions.
- The correct represntations.
- The Higgs field and the spontaneous symmetry breaking.
- Why the Higgs mass, the fermion masses.
- A relation among the fermions and W mass

$$\sum_{\text{generations}} m_e^2 + m_\nu^2 + 3m_d^2 + 3m_u^2 = 8M_W^2.$$

  a top quark mass of the order of $\frac{1}{\sqrt{2}} y_0\, v \sim$ 173.683 y_0 Gev.

The see-saw mechanism, however, suggests that the Yukawa coupling for the τ neutrino is of the same order as the top quark Yukawa coupling. Indeed, even if the tau neutrino mass has an upper bound of the order

$$m_{\nu_\tau} \leq 18.2 \text{ MeV},$$

the see-saw mechanism allows for a large Yukawa coupling term and in effect lowering the value of y_0 to $y_0 \sim 1.04$, which yields an acceptable value for the top quark mass of $179 - 173$ Gev depending on unification scale.

There are two predictions that do not fit with experiment, both related to the nature of the function f in the spectral action. The normalization of the kinetic terms imposes a relation between the coupling constants g_1, g_2, g_3 and the coefficient f_0, of the form

$$\frac{g_3^2\, f_0}{2\pi^2} = \frac{1}{4}, \qquad g_3^2 = g_2^2 = \frac{5}{3}\, g_1^2\,.$$

This gives that $\sin^2\theta_W = \frac{3}{8}$ a value also obtained in $SU(5)$ and $SO(10)$ grand unified theories.

The other prediction is that the Higgs quartic coupling is proportional to the the gauge couplings: $\lambda{=}\frac{4}{3+n} g_3^2$ where $n = 0, 1$ giving a Higgs mass of the order of $180 - 160$ Gev. which have been ruled out experimentaly.

Higher order corrections play an important role in the effective spectral action. A perturbative expansion reveals that the Higgs field could be rescaled so that the Higgs potential takes the form

$$V = \Lambda^4 \left(\sum_{n=0}^{\infty} c_n \left(\frac{\overline{H}H}{\Lambda^2} \right)^n \right) \tag{2}$$

where c_n, $n \geq 2$ are related to derivatives of the function $f^{(n-2)}(0)$. Thus the vev of $\frac{H}{\Lambda}$ is of order one, and it is questionable to truncate the potential to quartic terms in evaluating the minimum. In addition if one looks at the higher order interactions of the Higgs fields with the gauge fields one finds terms of the form

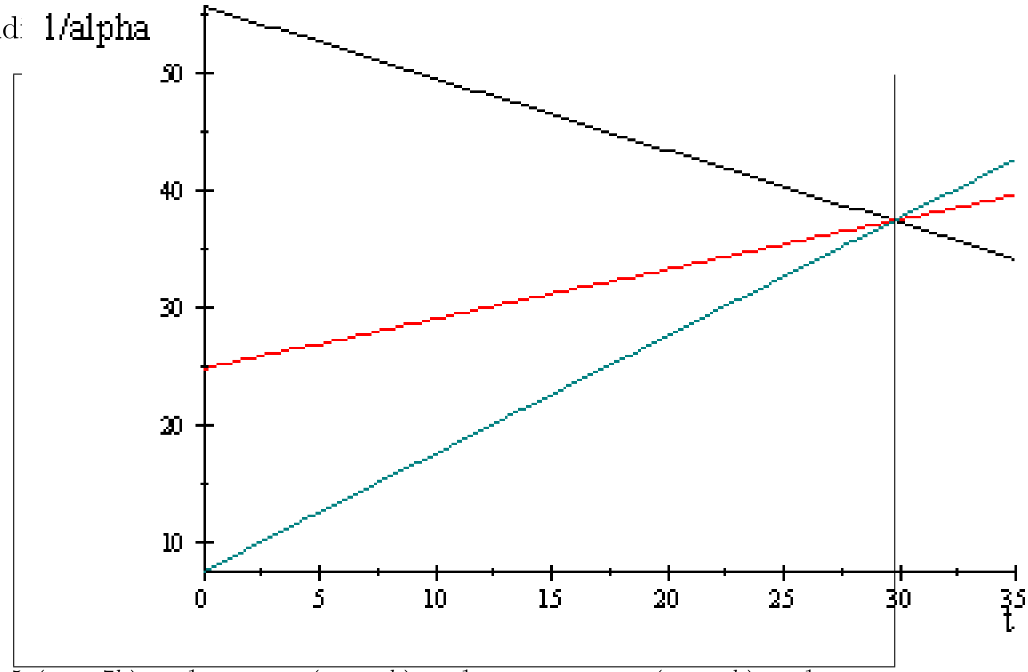
$$\sum_{n=0}^{\infty} c'_{2n} F^m_{\mu\nu} F^{\mu\nu m} \left(\frac{\overline{H}H}{\Lambda^2} \right)^n \tag{3}$$

where $F^m_{\mu\nu}$ are the curvatures of the gauge fields and c'_n, $n \geq 0$ are related to derivatives of the function $f^{(n)}(0)$. Thus when the Higgs field is expanded around its vev $v \sim \Lambda$, one finds that terms of all orders contribute to the kinetic term $F^m_{\mu\nu} F^{\mu\nu m}$. We deduce that although the cutoff function which vanishes beyond the scale Λ is a good approximation to reality, it does differ from the actual function, and this difference is responsible for the observed deviation. Even if one assumes that the spectral function is such that the series expansion converges rapidly, it will still be necessary to compute the first few terms in the heat kernel expansion beyond a_4 such as a_6 of order $\frac{1}{\Lambda^2}$and a_8 of order $\frac{1}{\Lambda^4}$. Unfortunately this is no small feat, and even for a_6 there are thousands of terms, as the Dirac operator of the Standard model is 384×384 matrix. It is then essential to determine the effects of expanding the Higgs field around its vev without the need to compute all the higher order terms explicitly, which is an impossible task. What is needed is a method to sum in a non-perturbative way the Higgs related higher order corrections.

The key observation first made when computing the spectral action on the product of spheres, is that we can derive, under certain approximations, the non-perturbative contributions of the function when the Dirac operator is shifted by a constant matrix. Combining this with a perturbative treatment of the linear terms, we can find the total effects of the expansion around the Higgs vev on the effective spectral action. What we found is that the formula for gauge coupling unification gets corrections of the form

$$\frac{5}{3} \left(1 + \frac{7k - 9q}{60} \right) g_1^2 = \left(1 + \frac{k+q}{4} \right) g_2^2 = \left(1 + \frac{k}{6} \right) g_3^2 \tag{4}$$

where k depends on the first derivative and q on second derivative of the spectral function evaluated at the minimum of the potential. Remarkably, this relation admits a fixed unification scale at 7.4837×10^{14}Gev, independent of the parameters of the theory, for which running the RG equations down is consistent with experimental data. In addition, the effective spectral action will have a modified Higgs quartic coupling, which depends on a new parameter, which can accommodate a Higgs mass as low as 135 Gev. To accommodate lower values one has, as for the commutative standard model, allow for the running of the quartic Higgs coupling to become negative at very h
includ



$\frac{5}{3}\left(1+\frac{7k}{60}\right)\alpha_1^{-1}$(black) $\left(1 \ +\frac{k}{4}\right)\alpha_2^{-1}$ (dark grey) $\left(1 \ +\frac{k}{6}\right)\alpha_3^{-1}$ (light grey),$k=1.05$

5 Spectral Action for NC Spaces with Boundary

In the Hamiltonian quantization of gravity it is essential to include boundary terms in the action as this allows to define consistently the momentum conjugate to the metric. This makes it necessary to modify the Einstein-Hilbert action by adding to it a surface integral term so that the variation of the action is well defined. The reason for this is that the curvature scalar R contains second derivatives of the metric, which are removed after integrating by parts to obtain an action which is quadratic in first derivatives of the metric. To see this note that the curvature $R\sim\partial\Gamma+\Gamma\Gamma$ where $\Gamma\sim g^{-1}\partial g$ has two parts, one part is of second order in derivatives of the form $g^{-1}\partial^{2}g$ and the second part is the square of derivative terms of the form $\partial g\partial g.$ To define the conjugate momenta in the Hamiltonian formalism, it is necessary to integrate by parts the term $g^{-1}\partial^{2}g$ and change it to the form $\partial g\partial g$. These surface terms, which turned out to be very important, are canceled by modifying the Euclidean action to

$$I=-\frac{1}{16\pi}\int\limits_{M}d^{4}x\sqrt{g}R-\frac{1}{8\pi}\int\limits_{\partial M}d^{3}x\sqrt{h}K,$$

where ∂M is the boundary of $M,$ h_{ab} is the induced metric on ∂M and K is the trace of the second fundamental form on ∂M. Notice that there is a relative factor of 2 between the two terms, and that the boundary term has to be completely fixed. This is a delicate fine tuning and is not determined by any symmetry, but only by the consistency requirement. There is no known symmetry that predicts this combination and it is always added by hand. In contrast we can compute the spectral action for manifolds with boundary. The hermiticity of the Dirac operator

$$\left(\psi|\,D\psi\right)=\left(D\psi|\,\psi\right)$$

is satisfied provided that $\pi_{-}\psi|_{\partial M}=0$ where $\pi_{-}=\frac{1}{2}\left(1-\chi\right)$ is a projection operator on ∂M with $\chi^{2}=1$. To compute the spectral action for manifolds with boundary we have to specify the condition $\pi_{-}D\psi|_{\partial M}=0.$ The result of the computation gives the remarkable result that the Gibbons-Hawking boundary term is generated without any fine tuning. Adding matter interactions, does not alter the relative sign and coefficients of these two terms, even when higher orders are included. The Dirac operator for a product space such as that of the standard model, must now be taken to be of the form

$$D = D_1 \otimes \gamma_F + 1 \otimes D_F$$

instead of

$$D = D_1 \otimes 1 + \gamma_5 \otimes D_F$$

because γ_5 does not anticommute with D_1 on ∂M.

The important point in the above result is the emergence of the combination

$$-\int_M d^4x\sqrt{g}R - 2\int_{\partial M} d^3x\sqrt{h}K$$

as the lowest term of the gravitational action which is known to be the required correction to the Einstein action involving the surface term so as to make the Hamiltonian formalism consistent. This is remarkable because both the sign and the coefficient are correct. The only assumption made is that normal boundary conditions are taken such that they enforce the hermiticity of the Dirac operator. This is yet another miracle concerning correct signs obtained in the spectral action of the Dirac operator. We also notice that the relative coefficient between R and K depends, in general, on the nature of the Laplacian. The desired answer is true for the square of the Dirac operator, but *not* for a general Laplacian. We note that there other boundary conditons may lead to different results.

5.1 Dilaton as the Dynamical Scale

Replacing the cutoff scale Λ in the spectral action, replacing $f(\frac{D^2}{\Lambda^2})$ by $f(P)$ where $P = e^{-\phi}D^2e^{-\phi}$ modifies the spectral action with dilaton dependence to the form

$$\text{Tr } F(P) \simeq \sum_{n=0}^{6} f_{4-n}\int d^4x\sqrt{g}e^{(4-n)\phi}a_n\left(x, D^2\right)$$

One can then show that the dilaton dependence almost disappears from the action if one rescales the fields according to

$$G_{\mu\nu} = e^{2\phi} g_{\mu\nu}$$
$$H' = e^{-\phi} H$$
$$\psi' = e^{-\frac{3}{2}\phi} \psi$$

With this rescaling one finds the result that the spectral action is

$$I\left(g_{\mu\nu} \to G_{\mu\nu}, H \to H', \psi \to \psi'\right) + \frac{24 f_2}{\pi 2} \int d^4x \sqrt{G} G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

scale invariant (independent of the dilaton field) except for the kinetic energy of the dilaton field ϕ. The dilaton field has no potential at the classical level. It acquires a Coleman-Weinberg potential through quantum corrections, and thus a vev. The dilaton acquires a very small mass. The Higgs sector in this case becomes identical with the Randall-Sundrum model. In that model there are two branes in a five dimensional space, one located at $x_5 = 0$ representing the invisible sector, and another located at $x_5 = \pi r_c$, the visible sector. The physical masses are set by the symmetry breaking scale $v = v_0 e^{-k r_c \pi}$ so that $m = m_0 e^{-k r_c \pi}$. If the bare symmetry breaking scale is taken at $m_0 \sim 10^{19}$ Gev, then by taking $k r_c \pi = 10$ one gets the low-energy mass scale $m \sim 10^2$ Gev. It is not surprising that the Randall-Sundrum scenario is naturally incorporated in the noncommutative geometric model, because intuitively one can think of the discrete space as providing the different brane sectors.

One immediate application for this is cosmology. All the features required by the inflationary model of Shaposhnikov et al based on the Standard model, are already present in the noncommutative construction. The main advantage is that the structure of all terms are completely determined by the spectral function, $f(D)$. For example the potential including higher order corrections is given by

$$V_0(\overline{H}H) = \frac{1}{16\pi^2} \int d^4x \sqrt{g} \mathrm{Tr} \left(\int_{\overline{H}H[x]}^{\infty} (w - \overline{H}H\,[x]) f(w)\, dw \right) \qquad (5)$$

Similarly the interaction terms with curvature and gauge fields are all determined.

6 Conclusions

The noncommutative picture of space-time gives almost uniquely and in a predictive way the physics of the Standard model, when combined with the spectral action principle. The prediction that at low energies the only spectrum is that of the Standard model (plus a dilaton) is holding well experimentally. We have an example of Minimal Input, Maximal Outcome.

Having understood the approximate space-time we are now well prepared to push ahead and do further studies such as cosmology and develop a noncommutative space whose limit is the product space considered so far.